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On a set of singularly perturbed convection–diffusion equations

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Abstract

We study convergence properties of a first-order upwind difference scheme applied to a weakly coupled system of singularly perturbed convection–diffusion equations. We derive a priori and a posteriori error estimates that are robust with respect to the perturbation parameters. Thereby strengthening and generalising recent results (Appl. Numer. Math. 51 (2004) 171; in: A. Ansari, A Hegarty, G.I. Shishkin, Numerical Methods for Problems with Layer Phenomena, Limerick, 2004, pp. 33–39). The key ingredient of our analysis are strong negative-norm stability results obtained earlier by Andreev (Differential Equations 37(7) (2001) 923) and by Andreev and Kopteva (Differential Equations 34(7) (1998) 921)).

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1. Introduction

This article is prompted by a recent publication in [4,5] where the authors consider the following set of singularly perturbed convection–diffusion equations:

$$\mathscr{L}_1 u_1 := -\varepsilon_1 u_1'' - a_1 u_1' = f_1 \quad \text{in } (0,1), \qquad u_1(0) = u_1(1) = 0, \tag{1a}$$

$$\mathscr{L}_2 u_2 := -\varepsilon_2 u_2'' - a_2 u_2' = f_2 + c u_1' \quad \text{in } (0,1), \qquad u_2(0) = u_2(1) = 0, \tag{1b}$$

where $0 < \varepsilon_1, \varepsilon_2 \ll 1$ are small parameters and $a_\mu(x) \geqslant \alpha_\mu > 0$ for $x \in [0,1]$, $\mu = 1, 2$.

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The solution of (1a) has an exponential boundary layer of width $\mathcal{O}(\varepsilon_1)$ with the bounds

$$|u_1^{(k)}(x)| \leqslant C(1 + \varepsilon_1^{-k} \mathrm{e}^{-\alpha_1 x/\varepsilon_1}), \quad \text{see [6].} \tag{2}$$

Therefore, instead of the above system one could study the single equation

$$-\varepsilon_2 u_2'' - a_2 u_2' = f(\cdot, \varepsilon_1) \quad \text{with } |f^{(k)}(x, \varepsilon_1)| \leqslant C(1 + \varepsilon_1^{-(k+1)} \mathrm{e}^{-\alpha_1 x/\varepsilon_1}),$$

where here and throughout C denotes a generic constant, that is independent of the perturbation parameters ε_1 and ε_2 and of the number of degrees of freedom of any numerical method applied. However, as pointed out in [4], u_1' is not available and has to be replaced by an approximation obtained from the numerical solution of (1a) when solving for u_2. Thus, one has to study the effect of the discretisation error of the first equation on the numerical solution of the second one.

The aim of the present study is threefold. We like to give an analysis that is simpler than the one presented in [4], gives sharper error bounds and applies to more general discretisation meshes including Bakhvalov meshes.

The key to our simplified analysis is the negative-norm stability of Lemma 1 which was first established in [1,2]. The crucial point in the analysis in [4] is to bound the error in replacing u_1' in (1b) by a finite difference approximation of the numerical solution of (1a). In our analysis this can be avoided. Moreover, no so-called Shishkin decompositions of the u_μ are required.

An outline of the paper is as follows. In Section 2 we introduce the finite difference scheme, study its stability properties and derive a priori and a posteriori error estimates. Bounds for the solution of (1) and its derivatives are derived in Section 3 and then used to robust error estimates for Shishkin and Bakhvalov meshes.

Notation: Throughout C denotes a generic constant that is independent of the perturbation parameters ε_1 and ε_2 and of the number of mesh points used.

2. Discretisation

We discretise (1) by means of a first-order difference scheme on an arbitrary mesh $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ with local mesh size $h_i := x_i - x_{i-1}$ and maximal mesh size $h := \max_i h_i$.

Find $U_1, U_2 \in \mathbb{R}_0^{N+1} = \{v \in \mathbb{R}^{N+1} : v_0 = v_N = 0\}$ such that

$$[L_1 U_1]_i := -\varepsilon_1 U_{1,\bar{x}x;i} - a_{1,i} U_{1,x;i} = f_{1,i} \quad \text{for } i = 1, \ldots, N-1, \tag{3a}$$

$$[L_2 U_2]_i := -\varepsilon_2 U_{2,\bar{x}x;i} - a_{2,i} U_{2,x;i} = f_{2,i} + c_i U_{1,x;i} \quad \text{for } i = 1, \ldots, N-1 \tag{3b}$$

with

$$v_{x;i} := \frac{v_{i+1} - v_i}{h_{i+1}} \quad \text{and} \quad v_{\bar{x};i} := \frac{v_i - v_{i-1}}{h_i}.$$

2.1. Stability

For $v \in W^{1,\infty}$ we introduce the norms

$$\|v\|_\infty := \sup_{x \in (0,1)} |v(x)|, \quad \|v\|_{\varepsilon,\infty} := \varepsilon \|v'\|_\infty + \|v\|_\infty \quad \text{and} \quad \|v\|_{-1,\infty} := \min_{V \in : V' = v} \|V\|_\infty$$

and, for $v \in \mathbb{R}^{N+1}$, their discrete counterparts

$$\|v\|_{\infty,\omega} := \max_{k=0,\ldots,N} |v_k|, \quad |v|_{1,\infty,\omega} := \max_{k=0,\ldots,N-1} |v_{x;k}|,$$

$$\|v\|_{\varepsilon,\infty,\omega} := \varepsilon |v|_{1,\infty,\omega} + \|v\|_{\infty,\omega} \quad \text{and} \quad \|v\|_{-1,\infty,\omega} := \min_{V\in\mathbb{R}^{N+1}: V_x = v} \|V\|_{\infty,\omega}.$$

Lemma 1 (*Andreev [1], Andreev and Kopteva [2], Linß [10]*). *The operators $\mathscr{L}_\mu$ and L_μ, $\mu = 1, 2$, satisfy the stability inequalities*

$$\|v\|_{\varepsilon_\mu,\infty} \leqslant C \left\|\mathscr{L}_\mu v\right\|_{-1,\infty} \quad \textit{for all } v \in W_0^{1,\infty}$$

and

$$\|v\|_{\varepsilon_\mu,\infty,\omega} \leqslant C \|L_\mu v\|_{-1,\infty,\omega} \quad \textit{for all } v \in \mathbb{R}_0^{N+1}.$$

2.2. A priori error analysis

In this section, we use the stability of the discrete operators L_μ to establish a priori bounds for the error of the discretisation.

Theorem 1. *Let u_1 and u_2 be the solutions of* (1) *and U_1 and U_2 their approximation by* (3). *Then*

$$\|u_1 - U_1\|_{\varepsilon_1,\infty,\omega} \leqslant C \max_{k=0,\ldots,N-1} \int_{x_k}^{x_{k+1}} (1 + |u_1'(s)|)\, \mathrm{d}s \tag{4a}$$

and

$$\|u_2 - U_2\|_{\varepsilon_2,\infty,\omega} \leqslant C \max_{k=0,\ldots,N-1} \int_{x_k}^{x_{k+1}} (1 + |u_1'(s)| + |u_2'(s)|)\, \mathrm{d}s. \tag{4b}$$

Proof. The error estimate (4a) for the first component was given in [10] and is a simple combination of the analysis in [9] with Lemma 1.

In order to bound the error in the second component note that for any constant $\gamma \in \mathbb{R}$

$$L_2(u_2 - U_2) = -(\varepsilon_2 u_{2,\bar{x}} + a_2 u_2 + cU_1 + F + \gamma)_x,$$

where

$$F_i = \sum_{k=1}^{i-1} h_{k+1}(f_{2;k} - a_{2,x;k} u_{2;k} - c_{x;k} U_{1;k})$$

and

$$\varepsilon_2 u_2' + a_2 u_2 + c u_1 + \mathscr{F} = \text{const} \quad \text{with} \quad \mathscr{F}(x) = \int_0^x (f_2 - a_2' u_2 - c' u_1)(s)\, \mathrm{d}s$$

by (1b). Thus,

$$L_2(u_2 - U_2) = -(\varepsilon_2(u_{2,\bar{x}} - u_2') + c(U_1 - u_1) + F - \mathscr{F})_x.$$

Taylor expansions give

$$\left|h_{k+1} f_{2;k} - \int_{x_k}^{x_{k+1}} f_2(x)\,\mathrm{d}x\right| \leqslant \frac{h_{k+1}^2}{2}\,\|f_2'\|_\infty,$$

$$\left|h_{k+1} a_{2,x;k} u_{2;k} - \int_{x_k}^{x_{k+1}} (a_2' u_2)(x)\,\mathrm{d}x\right| \leqslant h_{k+1}\|a_2'\|_\infty \int_{x_k}^{x_{k+1}} |u_2'(s)|\,\mathrm{d}s,$$

$$\left|h_{k+1} c_{x;k} u_{1;k} - \int_{x_k}^{x_{k+1}} (c' u_1)(x)\,\mathrm{d}x\right| \leqslant h_{k+1}\|c'\|_\infty \int_{x_k}^{x_{k+1}} |u_1'(s)|\,\mathrm{d}s,$$

$$|h_{k+1} c_{x;k}(u_{1;k} - U_{1;k})| \leqslant h_{k+1}\|c'\|_\infty \|u_1 - U_1\|_{\infty,\omega}$$

and

$$\varepsilon_2\|u_{2,\bar{x}} - u_2'\|_k \leqslant \int_{x_{k-1}}^{x_k} |(a_2 u_2' + c u_1' + f_2)(s)|\,\mathrm{d}s$$

by (1b). To complete the proof combine these estimates with (4a) and apply Lemma 1. □

2.3. A posteriori error analysis

The stability of the continuous operators $\mathscr{L}_\mu$ can be employed to establish a posteriori bounds for the error of the discretisation. Immitating the analysis from [7,10] and combining it with some of the details of the previous section, we get the following result.

Theorem 2. *Let u_1 and u_2 be the solutions of* (1) *and U_1^I and U_2^I the piecewise linear interpolants of the numerical solution* (3). *Then*

$$\|u_1 - U_1^I\|_{\varepsilon_1,\infty} \leqslant C \max_{k=0,\dots,N-1} h_{k+1}(1 + |U_{1,x;k}|)$$

and

$$\|u_2 - U_2^I\|_{\varepsilon_2,\infty} \leqslant C \max_{k=0,\dots,N-1} h_{k+1}(1 + |U_{1,x;k}| + |U_{2,x;k}|).$$

3. Layer-adapted meshes

In order to correctly construct layer-adapted meshes precise a priori information on the exact solution is required. That information for u_1 is provided by (2), but how does u_2 behave?

First the operator $\mathscr{L}_2$ satisfies $\|v\|_\infty \leqslant C\|v\|_{L_1}$ with a constant C that is independent of ε_2. Hence

$$\|u_2\|_\infty \leqslant C\|f + c u_1'\|_{L_1} \leqslant C.$$

Furthermore, we have the representation

$$u_2(x) = \int_x^1 H(s)\,\mathrm{d}s + \kappa \int_x^1 \mathrm{e}^{-A(s)}\,\mathrm{d}s$$

with

$$A(x) = \frac{1}{\varepsilon_2} \int_0^x a_2(s)\,\mathrm{d}s, \qquad H(x) = \frac{1}{\varepsilon_2} \int_0^x (f_2 + cu_1')(s)\,\mathrm{e}^{A(s)-A(x)}\,\mathrm{d}s$$

and

$$\kappa = -\int_0^1 H(s)\,\mathrm{d}s \Big/ \int_0^1 \mathrm{e}^{-A(s)}\,\mathrm{d}s.$$

Thus,

$$u_2'(x) = -H(x) - \kappa \mathrm{e}^{-A(x)}. \tag{5}$$

Next (2) and $a_2 \geqslant \alpha_2$ imply

$$|H(x)| \leqslant C + \frac{C}{\varepsilon_1\varepsilon_2} \mathrm{e}^{-\alpha_2 x/\varepsilon_2} \int_0^x \mathrm{e}^{(\alpha_2/\varepsilon_2 - \alpha_1/\varepsilon_1)s}\,\mathrm{d}s \quad \text{for } x \in [0,1]. \tag{6}$$

For $\alpha_2\varepsilon_1 \neq \alpha_1\varepsilon_2$ we have

$$|H(x)| \leqslant C\left\{1 + \frac{1}{\varepsilon_1\alpha_2 - \varepsilon_2\alpha_1}(\mathrm{e}^{-\alpha_1 x/\varepsilon_1} - \mathrm{e}^{-\alpha_2 x/\varepsilon_2})\right\} \quad \text{for } x \in [0,1]. \tag{7}$$

Let $\gamma > 1$ be an arbitrary constant. Set $v := \varepsilon_1\alpha_2/\varepsilon_2\alpha_1$.
(i) If $v \geqslant \gamma$. Then

$$\alpha_1/\varepsilon_1 < \alpha_2/\varepsilon_2, \qquad \varepsilon_1\alpha_2 - \varepsilon_2\alpha_1 \geqslant \frac{\gamma-1}{\gamma}\,\varepsilon_1\alpha_2$$

and (7) yields

$$|H(x)| \leqslant C(1 + \varepsilon_1^{-1}\mathrm{e}^{-\alpha_1 x/\varepsilon_1}) \quad \text{for } x \in [0,1].$$

(ii) If $v \leqslant 1/\gamma$. Then

$$\alpha_2/\varepsilon_2 < \alpha_1/\varepsilon_1, \qquad \varepsilon_2\alpha_1 - \varepsilon_1\alpha_2 \geqslant \frac{\gamma-1}{\gamma}\,\varepsilon_2\alpha_1$$

and we get

$$|H(x)| \leqslant C(1 + \varepsilon_2^{-1}\mathrm{e}^{-\alpha_2 x/\varepsilon_2}) \quad \text{for } x \in [0,1].$$

(iii) Finally, consider $1/\gamma < v < \gamma$. Using $v < \gamma$, we get

$$|H(x)| \leqslant C\left(1 + \frac{1}{\varepsilon_1\varepsilon_2}\,\mathrm{e}^{-\alpha_2 x/\varepsilon_2} \int_0^x \mathrm{e}^{\alpha_1(\gamma-1)s/\varepsilon_1}\,\mathrm{d}s\right) \leqslant C\left(1 + \frac{1}{\varepsilon_2}\,\mathrm{e}^{-(\alpha_2/\varepsilon_2 - \alpha_1(\gamma-1)/\varepsilon_1)x}\right).$$

Next use $1/\gamma < v$

$$|H(x)| \leqslant C\left(1 + \frac{1}{\varepsilon_2}\,\mathrm{e}^{-\alpha_2[1-\gamma(\gamma-1)]x/\varepsilon_2}\right) \quad \text{for } x \in [0,1].$$

Note that $[1-\gamma(\gamma-1)] \nearrow 1$ for $\gamma \searrow 1$. Thus for any $\tilde{\alpha}_2 \in (0, \alpha_2)$, we have

$$|H(x)| \leqslant C\left(1+\varepsilon_2^{-1}\mathrm{e}^{-\tilde{\alpha}_2 x/\varepsilon_2}\right) \quad \text{for } x \in [0,1].$$

Combining our bounds on H for the various values of v, we get

$$|H(x)| \leqslant C(1+\varepsilon_1^{-1}\mathrm{e}^{-\alpha_1 x/\varepsilon_1}+\varepsilon_2^{-1}\mathrm{e}^{-\tilde{\alpha}_2 x/\varepsilon_2}) \quad \text{for } x \in [0,1] \tag{8}$$

by which we have bounded the first term in (5).

Integrate (8) to bound

$$\left|\int_0^1 H(s)\,\mathrm{d}s\right| \leqslant C.$$

Furthermore,

$$\int_0^1 \mathrm{e}^{-A(s)}\,\mathrm{d}s \geqslant \frac{\varepsilon_2}{\|a_2\|_\infty}.$$

Thus, $\kappa \leqslant C\varepsilon_2^{-1}$ and (5) yields

$$|u_2'(x)| \leqslant C(1+\varepsilon_1^{-1}\mathrm{e}^{-\alpha_1 x/\varepsilon_1}+\varepsilon_2^{-1}\mathrm{e}^{-\tilde{\alpha}_2 x/\varepsilon_2}) \quad \text{for } x \in [0,1], \tag{9}$$

since $\mathrm{e}^{-A(x)} \leqslant \mathrm{e}^{-\alpha_2 x/\varepsilon_2}$.

Recalling the results of Theorem 1, we get the error bound

$$\|u_\mu - U_\mu\|_{\varepsilon_\mu,\infty,\omega} \leqslant C \max_{k=0,\ldots,N-1} \int_{x_k}^{x_{k+1}} (1+\varepsilon_1^{-1}\mathrm{e}^{-\alpha_1 s/\varepsilon_1}+\varepsilon_2^{-1}\mathrm{e}^{-\tilde{\alpha}_2 s/\varepsilon_2})\,\mathrm{d}s, \quad \mu=1,2. \tag{10}$$

This can be used to immediately establish uniform convergence on Bakhvalov and Shishkin meshes and other layer-adapted meshes.

3.1. Bakhvalov meshes

A Bakhvalov mesh [3] for the numerical solution of (1) can be generated by equidistributing the function

$$M_{\mathrm{Ba}}(x) = \max\left\{1, \frac{K_1}{\varepsilon_1}\exp\left(-\frac{\alpha_1 x}{\sigma_1\varepsilon_1}\right), \frac{K_2}{\varepsilon_2}\exp\left(-\frac{\tilde{\alpha}_2 x}{\sigma_2\varepsilon_2}\right)\right\},$$

i.e., the mesh points are chosen such that

$$\int_{x_{i-1}}^{x_i} M_{\mathrm{Ba}}(x)\,\mathrm{d}x = N^{-1}\int_0^1 M_{\mathrm{Ba}}(x)\,\mathrm{d}x.$$

The quantities $K_i > 0$ determine the number of mesh points used to resolve the two overlapping layers, while the $\sigma_i > 0$ determine the grading of the mesh in the layer region.

Apply (10) in order to get

$$\|u_\mu - U_\mu\|_{\varepsilon_\mu,\infty,\omega} \leqslant CN^{-1} \quad \text{if } \sigma_1, \sigma_2 \geqslant 1.$$

For technical details the reader is referred to [8] or [11].

3.2. Shishkin meshes

Shishkin meshes are frequently studied. This is because of their simplicity—they are piecewise uniform. We describe a possible construction for (1). Let $q_i > 0$, $i = 1, \ldots, 3$, with $q_1 + q_2 + q_3 = 1$ and $\sigma_1, \sigma_2 > 0$ be mesh parameters. We set

$$\lambda_2 = \min\left\{q_1 + q_2, \max\left\{\frac{\varepsilon_1}{\alpha_1}, \frac{\varepsilon_2}{\tilde{\alpha}_2}\right\}\sigma_2 \ln N\right\}$$

and

$$\lambda_1 = \min\left\{\frac{q_1\lambda_2}{q_1 + q_2}, \min\left\{\frac{\varepsilon_1}{\alpha_1}, \frac{\varepsilon_2}{\tilde{\alpha}_2}\right\}\sigma_1 \ln N\right\}.$$

Then the subintervals $I_1 = [0, \lambda_1]$, $I_2 = [\lambda_1, \lambda_2]$ and $I_3 = [\lambda_2, 1]$ are divided into $q_i N$ equidistant subintervals (assuming that $q_i N$ are integers). A typical choice is to take $q_1 = q_2 = \frac{1}{4}$ and $q_3 = \frac{1}{2}$ and an $N > 0$ that is divisible by 4. The parameters q_1 and q_2 determine the amount of mesh points used to resolve the two boundary layers, while σ_1 and σ_2 determine the stretching of the mesh inside the layers.

Using (10) one can show that the error of the simple upwind scheme (3) on a Shishkin mesh satisfies

$$\|u_\mu - U_\mu\|_{\varepsilon_\mu,\infty,\omega} \leqslant C N^{-1} \ln N \quad \text{if } \sigma_1, \sigma_2 \geqslant 1,$$

cf. [8] or [11].

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